1.1 Boolean Algebra and Boolean Equations

Another method to formally describe the operation of a digital circuit is by using Boolean equations.

1.1.1 Boolean Algebra

George Boole, in 1854, developed a system of mathematical logic, which we now call **Boolean algebra**. Based on Boole's idea, Claude Shannon, in 1938, showed that circuits built with binary switches can be described easily using Boolean algebra. The abstraction from switches being off and on to the use of Boolean algebra is as follows. Let $B = \{0, 1\}$ be the Boolean algebra whose elements are one of the two values, 0 and 1. We define the operations AND (•), OR (+), and NOT (′) for the elements of B by the axioms in Figure **Error! No text of specified style in document.**.1(a). These axioms are simply the definitions as previously given in the truth tables for the AND, OR, and NOT operators.

A variable x is called a **Boolean variable** if x takes on only values in B (i.e., either 0 or 1). Consequently, we obtain the theorems in Figure **Error! No text of specified style in document.**.1(b) for single variable and Figure **Error! No text of specified style in document.**.1(c) for two and three variables.

1a.	$0 \bullet 0 = 0$	1b.	$1 + 1 = 1$
2a.	$1 \bullet 1 = 1$	2b.	$0 + 0 = 0$
3a.	$0 \bullet 1 = 1 \bullet 0 = 0$	3b.	$1 + 0 = 0 + 1 = 1$
4a.	$0' = 1$	4b.	$1' = 0$

(a)

5a.	$x \bullet 0 = 0$	5b.	$x + 1 = 1$	Null Element
6a.	$x \bullet 1 = 1 \bullet x = x$	6b.	$x + 0 = 0 + x = x$	Identity
7a.	$x \bullet x = x$	7b.	$x + x = x$	Idempotent
8a.	$(x')' = x$			Double Complement
9a.	$x \bullet x' = 0$	9b.	$x + x' = 1$	Inverse

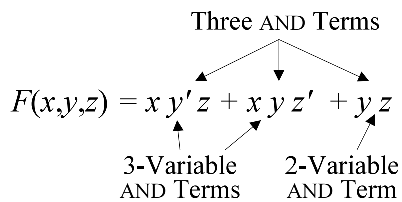
(b)

10a.	$x \bullet y = y \bullet x$	10b.	$x + y = y + x$	Commutative
11a.	$(x \bullet y) \bullet z = x \bullet (y \bullet z)$	11b.	$(x + y) + z = x + (y + z)$	Associative
12a.	$(x \bullet y) + (x \bullet z) = x \bullet (y + z)$	12b.	$(x + y) \bullet (x + z) = x + (y \bullet z)$	Distributive
13a.	$(x \bullet y)' = x' + y'$	13b.	$(x + y)' = x' \bullet y'$	DeMorgan's

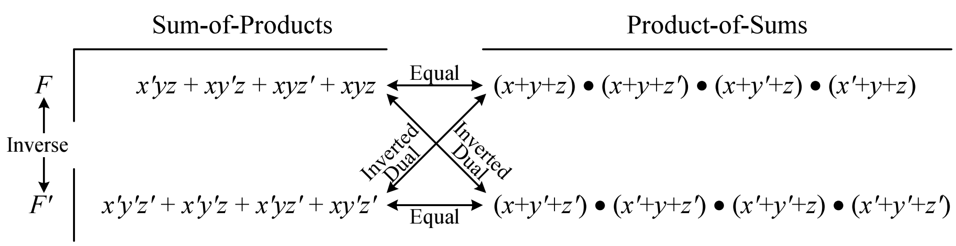
(c)

Figure Error! No text of specified style in document.**.1**. Boolean algebra axioms and theorems: (a) axioms; (b) single-variable theorems; (c) two- and three- variable theorems.

1.1.2 Boolean Functions and Their Inverses



x	y	z	F	F'
0	0	0	0	1
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	0

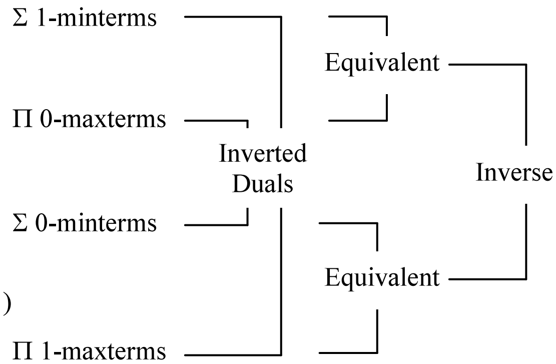


x	y	z	Minterm	Notation
0	0	0	$x'y'z'$	m_0
0	0	1	$x'y'z$	m_1
0	1	0	$x'yz'$	m_2
0	1	1	$x'yz$	m_3
1	0	0	$xy'z'$	m_4
1	0	1	$xy'z$	m_5
1	1	0	xyz'	m_6
1	1	1	xyz	m_7

x	y	z	Maxterm	Notation
0	0	0	$x + y + z$	M_0
0	0	1	$x + y + z'$	M_1
0	1	0	$x + y' + z$	M_2
0	1	1	$x + y' + z'$	M_3
1	0	0	$x' + y + z$	M_4
1	0	1	$x' + y + z'$	M_5
1	1	0	$x' + y' + z$	M_6
1	1	1	$x' + y' + z'$	M_7

The following summarizes these relationships for the function $F = xy'z + xyz' + yz$ and its inverse. Comparing these equations with those in **Error! Reference source not found.**, we see that they are identical.

$$
\begin{aligned}
F(x, y, z) &= x'yz + xy'z + xyz' + xyz \\
&= m_3 + m_5 + m_6 + m_7 \\
&= \Sigma(3, 5, 6, 7) \\
&= (x+y+z) \bullet (x+y+z') \bullet (x+y'+z) \bullet (x'+y+z) \\
&= M_0 \bullet M_1 \bullet M_2 \bullet M_4 \\
&= \Pi(0, 1, 2, 4) \\
F'(x, y, z) &= x'y'z' + x'y'z + x'yz' + xy'z' \\
&= m_0 + m_1 + m_2 + m_4 \\
&= \Sigma(0, 1, 2, 4) \\
&= (x+y'+z') \bullet (x'+y+z') \bullet (x'+y'+z) \bullet (x'+y'+z') \\
&= M_3 \bullet M_5 \bullet M_6 \bullet M_7 \\
&= \Pi(3, 5, 6, 7)
\end{aligned}
$$



Problems

2.11. Derive the truth table for the following Boolean functions.

a) $F(x,y,z) = x'y'z' + x'yz + xy'z' + xyz$
b) $F(x,y,z) = xy'z + x'yz' + xyz + xyz'$
c) $F(w,x,y,z) = w'xy'z + w'xyz + wxy'z + wxyz$
d) $F(w,x,y,z) = wxy'z + w'yz' + wxz + xyz'$
e) $F(x,y,z) = xy' + x'y'z + xyz'$
f) $F(w,x,y,z) = w'z' + w'xy + wx'z + wxyz$
g) $F(x,y,z) = [(x+y')\,(yz)']\,(xy' + x'y)$
h) $F(N_3,N_2,N_1,N_0) = N_3'N_2'N_1N_0' + N_3'N_2'N_1N_0 + N_3N_2'N_1N_0' + N_3N_2'N_1N_0 + N_3N_2N_1'N_0' + N_3N_2N_1N_0$

2.12. Derive the Boolean function for the following truth tables.

a)

a	*b*	*c*	*F*
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	0

b)

w	*x*	*y*	*z*	*F*
0	0	0	0	0
0	0	0	1	0
0	0	1	0	1
0	0	1	1	0
0	1	0	0	1
0	1	0	1	1
0	1	1	0	0
0	1	1	1	1
1	0	0	0	0
1	0	0	1	1
1	0	1	0	1
1	0	1	1	0
1	1	0	0	1
1	1	0	1	1
1	1	1	0	0
1	1	1	1	1

c)

w	*x*	*y*	*z*	F_1	F_2
0	0	0	0	1	1
0	0	0	1	0	1
0	0	1	0	0	1
0	0	1	1	1	1
0	1	0	0	0	0
0	1	0	1	1	1
0	1	1	0	1	0
0	1	1	1	0	0
1	0	0	0	0	1
1	0	0	1	1	1
1	0	1	0	1	0
1	0	1	1	0	0
1	1	0	0	1	1
1	1	0	1	0	1
1	1	1	0	0	1
1	1	1	1	1	1

d)

N_3	N_2	N_1	N_0	*F*
0	0	0	0	0
0	0	0	1	0
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	0
0	1	1	0	1
0	1	1	1	0
1	0	0	0	0
1	0	0	1	0
1	0	1	0	1
1	0	1	1	1
1	1	0	0	1
1	1	0	1	0
1	1	1	0	0
1	1	1	1	1

2.13. Use a truth table to show that the following variations of the DeMorgan's Theorem are true.

a) $(x + y)' = x' \bullet y'$
b) $(x + y + z)' = x' \bullet y' \bullet z'$
c) $(x \bullet y \bullet z)' = x' + y' + z'$
d) $(w \bullet x \bullet y \bullet z)' = w' + x' + y' + z'$

2.14. Use a truth table to show that the following equations are true.
a) $w'z' + w'xy + wx'z + wxyz = w'z' + xyz + wx'y'z + wyz$
b) $z + y' + yz' = 1$
c) $xy'z' + x' + xyz' = x' + z'$
d) $xy + x'z + yz = xy + x'z$
e) $w'x'yz' + w'x'yz + wx'yz' + wx'yz + wxyz = y(x' + wz)$
f) $w'xy'z + w'xyz + wxy'z + wxyz = xz$
g) $x_iy_i + c_i(x_i + y_i) = x_iy_ic_i + x_iy_ic_i' + x_iy_i'c_i + x_i'y_ic_i$
h) $x_iy_i + c_i(x_i + y_i) = x_iy_i + c_i(x_i \oplus y_i)$

2.15. Use Boolean algebra to show that $x \bullet (x + y) = x$ is true.

2.16. Use Boolean algebra to show that $x + (x \bullet y) = x$ is true.

2.17. Use Boolean algebra to show that $(x \bullet y) + (x \bullet y') = x$ is true.

2.18. Use Boolean algebra to show that $(x + y) \bullet (x + y') = x$ is true.

2.19. Use Boolean algebra to show that the equations in Problem 2.14 are true.

2.20. Use Boolean algebra to reduce the functions in Problem 1 as much as possible.

2.21. Use Boolean algebra to reduce the equation $F(x,y,z) = (x' + y' + x'y' + xy)\ (x' + yz)$ as much as possible.

2.22. Any function can be implemented directly either as specified or as its inverted form with a NOT gate added at the final output. Assume that the circuit size is proportional to only the number of AND gates and OR gates (i.e., ignore the number of NOT gates in determining the circuit size). Determine which form of the function (the inverted or non-inverted) will result in a smaller circuit size for the following function. Give your reason, and specify how many AND and OR gates are needed to implement the smaller circuit.

$$F(x,y,z) = x'y'z' + x'y'z + xy'z + xy'z' + xyz$$

2.23. Derive the truth table for the following logic gates.
a) A 4-input AND gate.
b) A 4-input NAND gate.
c) A 4-input NOR gate.
d) A 4-input XOR gate.
e) A 4-input XNOR gate.
f) A 5-input XOR gate.
g) A 5-input XNOR gate.

2.24. Derive the truth table for the following Boolean functions.
a) $F(w,x,y,z) = [(x \odot y)' + (xyz)']\ (w' + x + z)$
b) $F(x,y,z) = x \oplus y \oplus \mathrm{z}$
c) $F(w,x,y,z) = [w'xy'z + w'z\ (y \oplus x)]'$

2.25. Use Boolean algebra to convert the functions in Problem 2.24 to:
a) The sum-of-minterms format
b) The product-of-maxterms format

2.26. Use Boolean algebra to reduce the functions in Problem 2.24 as much as possible.

2.27. Use a truth table to show that the following equations are true.

a) $(x \oplus y) = (x \odot y)'$

b) $x \oplus y' = x \odot y$

c) $(w \oplus x) \odot (y \oplus z) = (w \odot x) \odot (y \odot z) = (((w \odot x) \odot y) \odot z)$

d) $[((xy)'x)' ((xy)'y)']' = x \oplus y$

2.28. Use Boolean algebra to show that the equations in Problem 2.27 are true.

2.29. Use Boolean algebra to show that $x \oplus y \oplus z = x'y'z + x'yz' + xy'z' + xyz$.

2.30. Use Boolean algebra to show that XOR = XNOR for three inputs.

2.31. Express the Boolean functions in Problem 1 using:

a) The Σ notation

b) The Π notation

2.32. Write the following equations as a Boolean function in the canonical form.

a) $F(x, y, z) = \Sigma(1, 3, 7)$

b) $F(w, x, y, z) = \Sigma(1, 3, 7)$

c) $F(x, y, z) = \Pi(1, 3, 7)$

d) $F(w, x, y, z) = \Pi(1, 3, 7)$

e) $F' (x, y, z) = \Sigma(1, 3, 7)$

f) $F' (x, y, z) = \Pi(1, 3, 7)$

2.33. Given $F' (x, y, z) = \Sigma(1, 3, 7)$, express the function F using a truth table.

2.34. Use Boolean algebra to convert the function $F(x, y, z) = \Sigma(3, 4, 5)$ to its equivalent product-of-sums canonical form.

2.35. Given $F = xy'z' + xy'z + xyz' + xyz$, write the equation for F' using:

a) The product-of-sums format

b) The sum-of-products format

2.36. Use Boolean algebra to convert the equation $F = w \odot x \odot y \odot z$ to:

a) The sum-of-minterms format

b) The product-of-maxterms format

2.37. Write the complete dataflow Verilog code for the Boolean functions in Problem 2.24.

2.38. Write the complete dataflow VHDL code for the Boolean functions in Problem 2.24.

2.39. Write the complete behavioral Verilog code for the car security system circuit discussed in Section **Error! Reference source not found.**.

2.40. Write the complete behavioral VHDL code for the car security system circuit discussed in Section **Error! Reference source not found.**.

